Course 4

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MCM

$$E \begin{cases} \nearrow [E:\mathbb{Q}_p] < +\infty \\ \searrow \mathbb{F}_q((\pi)) \end{cases}$$

$F|\mathbb{F}_q$ perfectoid. $\mathbb{A} = \begin{cases} W_{O_E}(O_F) \\ O_F[[\pi]] \end{cases}$

Recall: $Y = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(\pi[\varpi])$

$B = \mathcal{O}(Y)$

Define $\underbrace{|Y|^{cl}}_{\text{classical Tate points}} \subset |Y| =$ locally spectral top. space

Def: $\xi = \sum_{n \geq 0} [x_n] \pi^n \in \mathbb{A}$ is primitive if $x_0 \neq 0$,

$x_0, \dots, x_{d-1} \in m_F$ and $x_d \in O_F^\times$.

$\deg(\xi) := d$. $\qquad \deg(\xi\xi') = \deg(\xi) + \deg(\xi')$

$\rightarrow$ notion of primitive irreducible

* ξ primitive irreducible

$\Rightarrow V(\xi) \subset Y$ is one closed point y s.t.

$$k(y) = A[\tfrac{1}{\pi}]/\xi \quad \text{is perfectoid} \mid E$$

$$\mathcal{O}_{k(y)} = A/\xi \overset{\shortparallel}{=} B/\xi$$

and $[k(y)^\flat : F] = d$.

* $|Y|^{cl} = \mathrm{Prim}^{\mathrm{irred}}/A^\times = \{V(\xi) \,/\, \xi \text{ prim. irred.}\}$

$$\cap$$

$$|Y|$$

* For $y \in |Y|^{cl}$ $\quad \widehat{\mathcal{O}}_{Y,y} = \xi\text{-adic Completion of } B$

$= \text{"} \quad \text{"} \; A[\tfrac{1}{\pi}]$

$= B^+_{dR,y}(\cancel{\ \ })$

$=$ D.V.R. residue field $k(y)$

uniformizing element ξ

$$\mathrm{ord}_y : B \longrightarrow \mathbb{N} \cup \{+\infty\}$$

* F alg. closed $\Rightarrow \forall \xi$ primitive

$$\xi = u \times (\pi - [a_1]) \times \cdots \times (\pi - [a_d])$$

$u \in A^\times \qquad a_1, \ldots, a_d \in \mathfrak{m}_F \setminus \{0\}$

(Weierstrass factorization)

i.e. primitive irreducible $\Leftrightarrow$ deg $= 1$

and then $\forall y \in |Y|^{cl}$, $k(y)|E$ is alg. closed

$*$ $|\cdot| : |Y|^{cl} \longrightarrow]0,1[$, $y \in |Y|^{cl}$, $y = V(\xi)$, $\xi = \sum_{n \geq 0} [x_n]\pi^n$

$y \longmapsto |\pi(y)|$ $\qquad |y| = |x_0|^{1/\deg \xi}$

$Y_I = \mathrm{Int}(\{y \in Y \,/\, |\pi(y)| \in I\})$ for $I \subset]0,1[$ interval

$B_I = \mathcal{O}(Y_I)$

[Th: I compact $\Rightarrow$ B_I is a P.I.D. with $\mathrm{Spm}(B_I) = |Y_I|^{cl}$]

Localization of zeros: $f \in B \setminus \{0\}$

[$\{$slopes of $\mathrm{Newt}(f)$ wt. mult.$\} = \{-\log_q |y|$ with multiplicity $\mathrm{ord}_y(f)\}$]

Then
$$\mathrm{div} : B \setminus \{0\} \longrightarrow \mathrm{Div}^+(Y)$$
$$f \longmapsto \sum_{y \in |Y|^{cl}} \mathrm{ord}_y(f)\,[y]$$

Induces an injection

$$\mathrm{div} : B\setminus\{0\}/B^\times \hookrightarrow \mathrm{Div}^+(Y)$$

satisfying $\mathrm{div}(f) \geq \mathrm{div}(g) \Leftrightarrow f \in Bg$.

Rem. In general not surjective. But for $\rho \in \,]0,1[$

$$B_{]0,\rho]}\setminus\{0\} \,/\, B^\times_{]0,\rho]} \xrightarrow{\sim} \mathrm{Div}^+(Y_{]0,\rho]})$$

$\Rightarrow$ $B_{]0,\rho]}$ is a Bezout ring

$\Rightarrow$ $R := \varinjlim_{\rho \to 0} B_{]0,\rho]}$ is Bezout

Parametrization of $|Y|^{cl}$:

$E = \mathbb{F}_q((t)) \Rightarrow |Y|^{cl} = |\mathbb{D}_F^*|^{cl} = m_F \setminus \{0\}$

F alg. closed :

$$\begin{aligned} m_F \setminus \{0\} &\xrightarrow{\sim} |Y|^{cl} \\ a &\longmapsto V(\pi - a) \end{aligned}$$

But for $E|\mathbb{Q}_p$ any $y \in |Y|^{cl}$ can be written as $y = V(\pi - [a])$ with $a \in m_F \setminus \{0\}$ non unique i.e.

$$\begin{aligned} m_F \setminus \{0\} &\twoheadrightarrow |Y|^{cl} \\ a &\longmapsto V(\pi - [a]) \end{aligned}$$

surjective non-injective.

Solution: $E = \mathbb{Q}_p$. (Same with any E using L.T. groups)

$\widehat{\mathbb{G}}_m(\mathcal{O}_F) = (1 + m_F, \times) = \mathbb{Q}_p$-vector space (even Banach space)

In fact this is a $\mathbb{Z}_p$-module via $\forall a \in \mathbb{Z}_p$, $\forall \varepsilon \in 1 + m_F$

$$a.\varepsilon = a.\left(1+(\varepsilon-1)\right) = \sum_{b\geq 0}\binom{a}{b}(\varepsilon-1)^b$$

and $p.\varepsilon=\varepsilon^p \Rightarrow p$ invertible

$\hookrightarrow$ F perfect

$$\Rightarrow (1+m_F, \times) = \mathbb{Q}_p\text{-v.s.}$$

Def: $\varepsilon\in 1+m_F \setminus \{1\}$

$\varepsilon\neq 1$

$$u_\varepsilon = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1+[\varepsilon^{1/p}]+\dots+[\varepsilon^{\frac{p-1}{p}}] \in A$$

"Gauss sum"

Lemma: u_ε is primitive of degree 1

$$\rightarrow u_\varepsilon \bmod \pi = 1+\varepsilon^{1/p}+\dots+\varepsilon^{\frac{p-1}{p}} = \frac{\varepsilon-1}{\varepsilon^{1/p}-1}\neq 0 \text{ since } \varepsilon\neq 1$$

$u_\varepsilon \bmod W(m_F) \in W(k_F)$

$\parallel$

$1+[1]+\dots+[1]=p$ since $\varepsilon\mapsto 1\in W(k_F)$ $\square$

For $\varepsilon \in 1+\mathfrak{m}_F \setminus \{1\}$ set $C_\varepsilon = B/u_\varepsilon$

$$\vartheta_\varepsilon : B \twoheadrightarrow C_\varepsilon$$

$$\varepsilon \in F = C_\varepsilon^\flat \qquad \varepsilon^{(n)} \in C_\varepsilon$$

$$(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$$

$$\| $$

$$(\varepsilon^{(n)})_{n\geq 0} = \left(\vartheta_\varepsilon\left([\varepsilon^{1/p^n}]\right)\right)_{n\geq 0}$$

But $1+\varepsilon^{(1)}+\dots+(\varepsilon^{(1)})^{p-1} = \vartheta_\varepsilon\Big(\underbrace{1+[\varepsilon^{1/p}]+\dots+[\varepsilon^{\frac{p-1}{p}}]}_{u_\varepsilon}\Big) = 0$

$$\Rightarrow \boxed{\varepsilon^{(1)} \in \mu_p(C_\varepsilon)}$$

Moreover $\mathcal{O}_{C_\varepsilon}/p\,\mathcal{O}_{C_\varepsilon} = W(\mathcal{O}_F)/(p, u_\varepsilon)$

$$= \mathcal{O}_F/\overline{u_\varepsilon} \qquad \overline{u_\varepsilon} = \frac{\varepsilon-1}{\varepsilon^{1/p}-1}$$

$$(\varepsilon^{(1)}-1) \bmod p \equiv \varepsilon^{1/p}-1 \bmod \overline{u_\varepsilon}$$

But $|\varepsilon^{1/p}-1| = |\varepsilon-1|^{\frac{1}{p}} < |\overline{u_\varepsilon}| = |\varepsilon-1|^{1-\frac{1}{p}}$

$\Rightarrow \quad \varepsilon^{(1)} - 1 \neq 0$ i.e. $\varepsilon^{(1)} \neq 1$

$\Rightarrow \varepsilon^{(1)} \in \mu_p(C_\varepsilon)$ is a primitive root of unity.

$$\left[\Rightarrow \varepsilon = \left(\varepsilon^{(n)}\right)_{n \geq 0} = \text{base of } \underbrace{\mathbb{Z}_p(1)(C_\varepsilon)}_{T_p(\widehat{\mathbb{G}}_m) \text{ over } C_\varepsilon} \right]$$

~~Rk~~: $\rightarrow$ any $\varepsilon \in 1 + \mathfrak{m}_F \setminus \{1\}$ can be seen as the base of $T_p(\widehat{\mathbb{G}}_m)/C_y$ for some $y \in |Y|^{cl}$

$$T_p(\widehat{\mathbb{G}}_m)/C_y = \mathbb{Z}_p(1)(C_y)$$

<u>Prop:</u> The application $(1 + \mathfrak{m}_F) \setminus \{1\} \longrightarrow |Y|^{cl}$

$$\varepsilon \longmapsto V(u_\varepsilon)$$

induces a bijection $\boxed{(1 + \mathfrak{m}_F) \setminus \{1\} / \mathbb{Z}_p^\times \xrightarrow{\sim} |Y|^{cl}}$

$V = 1 + \mathfrak{m}_F = \mathbb{Q}_p$ - Banach via $\|\varepsilon\| = |\varepsilon - 1|$

$$\boxed{V \setminus \{0\} / \mathbb{Z}_p^\times \xrightarrow{\sim} |Y|^{cl}}$$

<u>Inverse given by:</u> $y \in |Y|^{cl}$

$C = k(y) \,|\, E$ alg. closed with $F = C^\flat$

$\varepsilon =$ base of $\mathbb{Z}_p(1)(C) \hookrightarrow (C^\flat)^\times$

$\rightsquigarrow C = C_\varepsilon$ and $y = V(u_\varepsilon)$.

Rem: In fact $|Y| = |Y^\diamond|$ and

$$Y^\diamond = \mathrm{Spa}(F) \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

$\mathrm{Spa}\, \mathbb{Q}_p^\diamond = \mathrm{Spa}(\mathbb{Q}_p^{cyc,\flat}) / \mathbb{Z}_p^\times$

Fontaine Wintenberger: $\mathbb{F}_p((T^{1/p^\infty}))$ with $T = \varepsilon - 1$, $\varepsilon = (\zeta_{p^n})_{n \geq 1}$

$$= \underbrace{\mathbb{D}_F^{*,1/p^\infty}}_{\text{perfectoid open punctured disk}} / \mathbb{Z}_p^\times$$

$$\Rightarrow \boxed{|Y| = |\mathbb{D}_F^*| / \mathbb{Z}_p^\times} \supset |Y|^{cl}$$

what we did is identify classical points.

Le cas F général

$G_F = \mathrm{Gal}(\overline{F}|F)$

Th: $|Y_{\widehat{\overline{F}}}|^{cl} \circlearrowleft G_F$

$|Y_{\widehat{\overline{F}}}|^{cl,\, G_F\text{-fini}} = \{ y \mid \#(G_F \cdot y) < +\infty \}$

There is a surjective G_F-invariant map

$$\beta : |Y_{\widehat{\overline{F}}}|^{cl,\, G_F\text{-fini}} \twoheadrightarrow |Y_F|$$

inducing $\boxed{|Y_{\widehat{\overline{F}}}|^{cl,\, G_F\text{-fini}} / G_F \xrightarrow{\sim} |Y_F|}$

i.e. fibers of β = Galois orbits.

Such that if $y = \beta(z)$ then $C_z | K_y$ and $C_z = \widehat{\overline{K_y}}$

alg. closed perfectoid

Moreover $\deg(y) = \#(G_F \cdot z)$

$$\text{and } \mathrm{Gal}(C_z | K_y) \xrightarrow{\sim} \underbrace{\mathrm{Aut}}_{\mathrm{Aut}\,\mathcal{C}^0}(C_z^b | K_y^b) = \mathrm{Gal}(\overline{\mathbb{F}} | K_y^b)$$

(purity theorem)

The curve

$\mathbb{A} \circlearrowleft \varphi$ Frobenius

given by $\sum_{n \geq 0} [x_n] \pi^n \longmapsto \sum_{n \geq 0} [x_n^q] \pi^n$

Continuous for the $(\pi, [\varpi])$-adic topology.

$$Y = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus \underbrace{V(\pi[\varpi])}_{\text{| divisor | stable under } \varphi}$$

$\circlearrowleft \varphi^{\mathbb{Z}}$

$$|\cdot| : |Y| \longrightarrow \,]0,1[$$
$$y \longmapsto |\pi(y)|$$

satisfies $|\varphi(y)| = |y|^{1/q}$

$\Rightarrow \varphi(\text{Annulus radius } \rho) = \text{Annulus radius } \rho^{1/q}$

$\Rightarrow \varphi^{\mathbb{Z}}$ acts properly discontinuously without fixed points.

$X^{ad} := Y/\varphi^{\mathbb{Z}}$ as an E-adic space. quasi-compact $\underline{\underline{\text{proper}}}$ (although not of finite type)

$\underline{\text{Ex}}$: $E = \mathbb{F}_q((\pi))$

$$\begin{tikzcd}
 & Y = \mathbb{D}_F^* \arrow[dl, "\text{not } \varphi \text{ invariant}"'] \arrow[dr, "\varphi\text{-invariant}"] & \\
\mathrm{Spa}(F) \arrow[loop below, "\varphi = \mathrm{Frob}"] & & \mathrm{Spa}(E)
\end{tikzcd}$$

$$X = \mathbb{D}_F^*/\varphi^{\mathbb{Z}} \longrightarrow \mathrm{Spa}(E)$$

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$$|X^{ad}| \longrightarrow \,]0,1[/ \varphi^{\mathbb{Z}} = S^1$$

fibered over the circle.

What is $Y / \varphi^{\mathbb{Z}}$?

Line bundles $/ X^{ad}$ = $\varphi^{\mathbb{Z}}$-equivariant line bundles $/ Y$

= φ-modules over B projective of rk. 1

$\simeq$ φ-modules over $\mathcal{R} = \varinjlim_{\rho \to 0} B_{]0,\rho]}$ free over $\mathcal{R}$ of rk. 1

$\underset{\text{easy}}{\Longrightarrow}$ $\mathrm{Pic}(X^{ad}) = \mathbb{Z}$

$n \mapsto (B.e, \varphi)$ with $\varphi(e) = \pi^{-n} e$

Define $\mathcal{O}(d) \longleftrightarrow (B, \pi^{-d}\varphi)$

Declare $\mathcal{O}(1)$ ample.

$$P = \bigoplus_{d \geq 0} \underbrace{H^0(X^{ad}, \mathcal{O}(d))}_{B^{\varphi = \pi^d} = \{f \in B \,/\, \varphi(f) = \pi^d f\}}$$

$\rightarrow X = \mathrm{Proj}(P)$ schematical curve.